Modes in Rectangular Guides Loaded with a Transversely Magnetized Slab of Ferrite away from the Side Walls*

G. BARZILAI†, SENIOR MEMBER, IRE, AND G. GEROSA†, ASSOCIATE, IRE

Summary—**The characteristic equation describing the general modal spectrum for a rectangular guide partially filled with a slab of ferrite transversely magnetized and situated away from the side walls is derived. This equation is numerically solved for particular cases and for modes of zero and first order with respect to the dependence along the direction of the dc magnetic field. Some experiments to verify the theoretical results are presented and show good agreement with the theory.**

I. Introduction

The authors in a previous paper[1] have discussed the modal spectrum in rectangular guides completely filled with transversely magnetized ferrite; in a successive paper[2] they have discussed the modal spectrum in rectangular guides loaded with a slab of transversely magnetized ferrite against one side wall.

The purpose of this work is to report on further theoretical study on a more general structure of a rectangular guide loaded with a slab of transversely magnetized ferrite away from the side walls and to describe some experiments carried out to verify the theory. The only analyses available up to date for this structure are, to our knowledge, relative to modes with no dependence along the direction of the dc magnetic field.[3–6]

The method of solution we have used is similar to the one already discussed,[1] but the derivation of the characteristic equation is very involved and the relative numerical computations, even by using a medium speed electronic computer, require considerable length of time. We have therefore limited our numerical investigation to some cases which we thought to be of interest.

II. The Characteristic Equation

Let us refer to the parallel plate guide represented in Fig. 1. We shall assume the walls to be perfectly conducting.

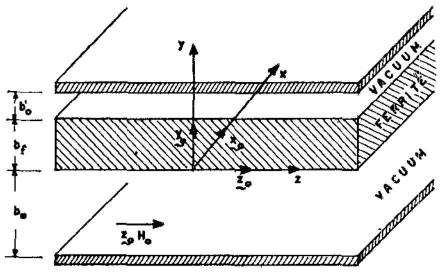

Fig. 1—Geometry of the parallel plate guide partially filled with magnetized ferrite.

Let us assume the dc magnetic field, of sufficient intensity H_0 to saturate the ferrite, to be directed along the z axis. We shall assume for the ferrite region a scalar dielectric constant $\epsilon_0\ \epsilon$ and a magnetic tensor permeability $\boldsymbol{\mu}$ given by the following expression (time dependence $\exp\ [j\omega t]$):

$$\boldsymbol{\mu} = \mu_0 \begin{vmatrix} \mu_1 & j\mu_2 & 0 \\ -j\mu_2 & \mu_1 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

where

$$\mu_1 = 1 + \frac{\rho}{1-\tau^2}; \quad \mu_2 = \frac{\tau\rho}{1-\tau^2}; \quad \rho = \frac{M_0}{\mu_0 H_0}; \quad \tau = \frac{\omega}{\omega_0},$$

and M_0 is the intensity of the saturation magnetization, ω and $\omega_0 = -\gamma H_0$ are the applied and the resonant circular frequencies, γ is the gyromagnetic ratio for the electron, μ_0 and ϵ_0 are the permeability and the dielectric constant of the vacuum.

As discussed in detail in our previous work,[1] to construct our modal solution we shall consider for the three

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† Istituto di Elettronica dell Università di Roma, Rome Italy.

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regions of width b_f, b_0 and b_0' fields having spatial dependence of the form:

$$\exp [j(k_x x + k_y y + k_z z)]. \quad (1)$$

We shall add to the propagation constants k_x, k_y, k_z a subscript f or 0 in order to refer to the ferrite or to the vacuum regions.

The propagation constants will be measured in units of $\omega \sqrt{\mu_0 \epsilon_0}$ and consequently lengths will be measured assuming as unit $1/\omega \sqrt{\mu_0 \epsilon_0}$.

In order to satisfy Maxwell's equations, we must have in the ferrite region

$$\mu_1 t^4 + [(\mu_1 + 1)k_{zf}^2 - \epsilon(\mu_1^2 - \mu_2^2 + \mu_1)]t^2 + k_{zf}^4 - 2\mu_1 \epsilon k_{zf}^2 + \epsilon^2(\mu_1^2 - \mu_2^2) = 0, \quad (2)$$

where

$$t^2 = k_{xf}^2 + k_{yf}^2, \quad (3)$$

and in the vacuum regions

$$k_{x0}^2 + k_{y0}^2 + k_{z0}^2 = 1. \quad (4)$$

Each modal solution of our problem is labelled by a pair of values k_z, k_x. In order to satisfy the boundary conditions at the interfaces we must have: $k_{zf} = k_{z0} = k_z$ and $k_{xf} = k_{x0} = k_x$.

For the ferrite region (2) yields, for a given k_z, two values of $t^2 : t_1^2$ and t_2^2. From (3) we obtain therefore two values of $k_{yf}^2 : k_{y1}^2$ and k_{y2}^2, for a given k_x. Consequently we express the field in the ferrite region as a superposition of four fields of the form (1) having the same k_x and k_z and arbitrary amplitudes.

In the vacuum regions for a given pair k_z, k_x we have only one value of k_{y0}^2, but, since a general field can be expressed as a superposition of TE and TM waves, for each vacuum region we shall have again four arbitrary amplitudes.

By imposing the tangential component of the electric field to be zero at the metallic boundaries and the tangential component of both the electric and magnetic fields to be continuous at the interfaces between vacuum and ferrite, we obtain a system of twelve linear algebraic homogeneous equations in the twelve unknown amplitudes. By setting equal to zero the determinant of the coefficients, we obtain the following characteristic equation:

$$\sum_{(i,k)} \sum_{(l,m)} (-1)^{i+k+l+m} Y_{(i,k)_c (l,m)_c}(b_f) \cdot X_{i,k}(b_0) X_{l,m}(-b_0') = 0, \quad (5)$$

where the index pairs i, k and l, m can assume the values 1,2; 1,3; 1,4; 2,3; 2,4; 3,4 and $(i, k)_c$ represents the pair which together with i, k complete the set of first four integral numbers; and similarly for $(l, m)_c$.

The functions X and Y have the following expressions:

$$X_{i,k}(b_0) = c_{i1}c_{k2} - c_{i2}c_{k1},$$

where

$$c_{11} = k_z(1 - k_x^2)k_{y0} \sin k_{y0}b_0 \qquad c_{12} = 0$$
$$c_{21} = - k_x k_z^2 k_{y0} \sin k_{y0}b_0 \qquad c_{22} = - k_{y0} \sin k_{y0}b_0$$
$$c_{31} = 0 \qquad c_{32} = - (1 - k_x^2) \cos k_{y0}b_0$$
$$c_{41} = - k_z k_{y0}^2 \cos k_{y0}b_0 \qquad c_{42} = k_x k_z \cos k_{y0}b_0$$

and

$$\begin{aligned} Y_{(i,k)(l,m)}(b_f) = & - k_{y1}k_{y2}(C_{ik,11}C_{lm,22} + C_{ik,22}C_{lm,11}) \\ & + \sin k_{y1}b_f \sin k_{y2}b_f(A_{ik,12}A_{lm,12} \\ & + k_{y1}^2 k_{y2}^2 B_{ik,12}B_{lm,12} + k_{y1}^2 C_{ik,21}C_{lm,21} \\ & + k_{y2}^2 C_{ik,12}C_{lm,12}) \\ & + k_{y1}k_{y2} \cos k_{y1}b_f \cos k_{y2}b_f(B_{ik,12}A_{lm,12} \\ & + A_{ik,12}B_{lm,12} + C_{ik,21}C_{lm,12} + C_{ik,12}C_{lm,21}) \\ & + k_{y1} \cos k_{y1}b_f \sin k_{y2}b_f[C_{ik,21}A_{lm,12} \\ & - A_{ik,12}C_{lm,21} + k_{y2}^2(C_{ik,12}B_{lm,12} \\ & - B_{ik,12}C_{lm,12})] \\ & + k_{y2} \sin k_{y1}b_f \cos k_{y2}b_f \cdot [A_{ik,12}C_{lm,12} \\ & - C_{ik,12}A_{lm,12} + k_{y1}^2(B_{ik,12}C_{lm,21} \\ & - C_{ik,21}B_{lm,12})], \end{aligned}$$

where

$$A_{ik,12} = a_{i1}a_{k2} - a_{k1}a_{i2}$$
$$B_{ik,12} = b_{i1}b_{k2} - b_{k1}b_{i2}$$
$$C_{ik,11} = a_{i1}b_{k1} - a_{k1}b_{i1}$$
$$C_{ik,22} = a_{i2}b_{k2} - a_{k2}b_{i2}$$
$$C_{ik,21} = a_{i2}b_{k1} - a_{k2}b_{i1}$$
$$C_{ik,12} = a_{i1}b_{k2} - a_{k1}b_{i2}$$

and similarly for l, m, and:

$$a_{11} = \mu_2 k_z(\epsilon - k_x^2) = a_{12}$$
$$b_{11} = - (\mu_1 - 1)k_z k_x = b_{12}$$
$$a_{21} = - \mu_2(\epsilon - t_1^2)k_x \qquad a_{22} = - \mu_2(\epsilon - t_2^2)k_x$$
$$b_{21} = k_z^2 - \mu_1(\epsilon - t_1^2) \qquad b_{22} = k_z^2 - \mu_1(\epsilon - t_2^2)$$
$$a_{31} = k_x^2[k_x^2 - (\epsilon - t_1^2)] - \epsilon[k_z^2 - \mu_1(\epsilon - t_1^2)] \qquad a_{32} = k_x^2[k_z^2 - (\epsilon - t_2^2)] - \epsilon[k_z^2 - \mu_1(\epsilon - t_2^2)]$$
$$b_{31} = 0 = b_{32}$$
$$a_{41} = - k_z(\mu_1\epsilon - k_z^2 - t_1^2)k_x \qquad a_{42} = - k_z(\mu_1\epsilon - k_z^2 - t_2^2)k_x$$
$$b_{41} = - k_z \mu_2 \epsilon = b_{42}.$$

It can be seen that (5) contains only even powers of k_z.

It can be verified that the solution possesses reflection symmetry along the z direction. Therefore a solution for a rectangular guide of height a can be obtained by choosing

$$k_z = \frac{m\pi}{a}, \qquad m = 0, 1, 2 \cdots \quad (6)$$

For $m=0$, (5) breaks down into two equations corresponding to TE and TM modes. TM modes cannot exist in the rectangular guide obtained by closing the structure of Fig. 1 with two metallic walls normal to the z-axis. TE modes can exist and the relative characteristic equation is the same as the one given by others. From $m\neq 0$, (5) does not in general break down into two equations and the resulting electromagnetic field cannot be resolved into TE and TM modes.

The solution does not possess reflection symmetry along the x axis. In addition, (5) contains in general odd powers of k_x, and the solution therefore is not reciprocal, *i.e.*, a solution $+k_x$ does not necessarily imply the solution $-k_x$.

It should be noted that obvious solutions of (5) are $k_{y1}=0$; $k_{y2}=0$; $k_{y0}=0$. These solutions however correspond to zero amplitude fields. It can be verified that

$$k_x^2 = \epsilon \frac{k_z^2 - \mu_1(\epsilon - t_{1,2}^2)}{k_z^2 - (\epsilon - t_{1,2}^2)}$$

are always solutions of (5), but these solutions also correspond to zero amplitude fields.

We note that by interchanging b_0 with b_0' and k_x with $-k_x$, (5) remains unchanged as it follows from symmetry considerations.

To discuss (5), it is convenient to find the asymptotic behavior of some of the solutions when $b_0'\to 0$. More exactly we shall assume b_f and b_0 finite and by letting $b_0'\to 0$ we shall look for solutions with real k_x such that $|k_x|\to\infty$. From (5) we obtain

$$\tanh |k_x| b_0' = \frac{\mu_2^2 - \mu_1(\mu_1+1) - \frac{k_x}{|k_x|}\mu_2}{\mu_1 + 1 - \frac{k_x}{|k_x|}\mu_2}. \qquad (7)$$

From (7) it is apparent that asymptotic solutions are possible only when the second member of the equation is positive. In such a case the asymptotic solutions are represented by hyperbolas. It should also be noted that (7) is independent of k_z and therefore the asymptotic solutions considered are the same for any k_z, *i.e.*, in the case of the rectangular guide for any m.

III. Numerical Analysis

There are several parameters which determine the solutions of (5), namely: quantities characteristic of the ferrite medium, *i.e.*, M_0 and ϵ; quantities describing the structure, *i.e.*, a, b, b_f and b_0' ($b_0=b-b_f-b_0'$); impressed quantities, *i.e.*, ω and H_0. The three quantities M_0, ω and H_0 enter our problem through the adimensional parameters ρ and τ, so that the actual parameters to be considered are ρ, ϵ, τ, a, b, b_f and b_0'.

When a set of these parameters has been chosen, (5) in virtue of (2), (3) and (4) becomes a relation between k_x and k_z^2. We shall fix k_z by means of (6), which allows us to label our modal solutions according to the integral values given to m. We shall call, therefore, modes of zero order those corresponding to $k_z=0$, and modes of first order those corresponding to $k_z=\pi/a$, etc.

For a given mode order, (5) determines the relative propagation constants k_x. In what follows we shall only look for real values of k_x, *i.e.*, for unattenuated propagating modes.

With respect to the field configuration in the cross section of the guide the mode order determines the z dependence. The y dependence is determined by the values of k_{y1}, k_{y2} and k_{y0}, which can assume real or purely imaginary values. In fact, by assuming k_z and k_x real, (4) states that k_{y0}^2 is real and (2), once solved with respect to t^2, shows that t_1^2 and t_2^2 are both real.

The solutions of (5) can be classified by dividing the field of variability of τ into six regions, exactly in the same manner as we have done in our previous work[2] relative to the slab of ferrite against one side wall.

We have numerically solved (5) for three cases which we thought to be of interest. Two of such cases are those corresponding to Figs. 5 and 6 of the quoted paper.[2]

The results of the numerical analysis are recorded in the diagram of Figs. 2–4, which show k_x vs b_0' for the values of the parameters indicated in the figures and for modes of zero and first order.

In the diagrams of Figs. 2–4 the various zones delimited by the straight lines $k_{y1}=0$, $k_{y2}=0$ and $k_{y0}=0$ have been shaded in different ways in order to recognize at a glance when k_{y1}, k_{y2}, and k_{y0} are real or imaginary. It is understood that when one of the three typical shadings indicated is present the relative k_y is real and when it is not it is imaginary. For instance, when no shading exists the three k_y's are all imaginary; when all three shadings are present the three k_y's are all real, and so on. It should be noted however that for modes of zero order k_{y1} is associated with TM modes, which have zero amplitude in the rectangular guide. The shading relative to k_{y1} has been therefore omitted in the diagrams relative to zero order modes.

The most interesting feature appearing from the diagrams is the fact that for suitable dimensions of the guide and of the ferrite slab unidirectional propagation exists for $b_0'=0$ and, as soon as the slab of ferrite is moved sufficiently away from the wall, for zero and first order modes no propagation can occur in either direction. It is reasonable to assume that higher order modes exhibit the same behavior. Experimental evidences described in the next section seem to agree with this theoretical result.

In Fig. 4 we have indicated with b_{0c}' the cutoff distance for modes of zero order. The cutoff distance for modes of first order is practically the same.

With reference to the zero order modes of Fig. 4 we have investigated the sign of the group velocity for the region where there are two propagating modes. We note

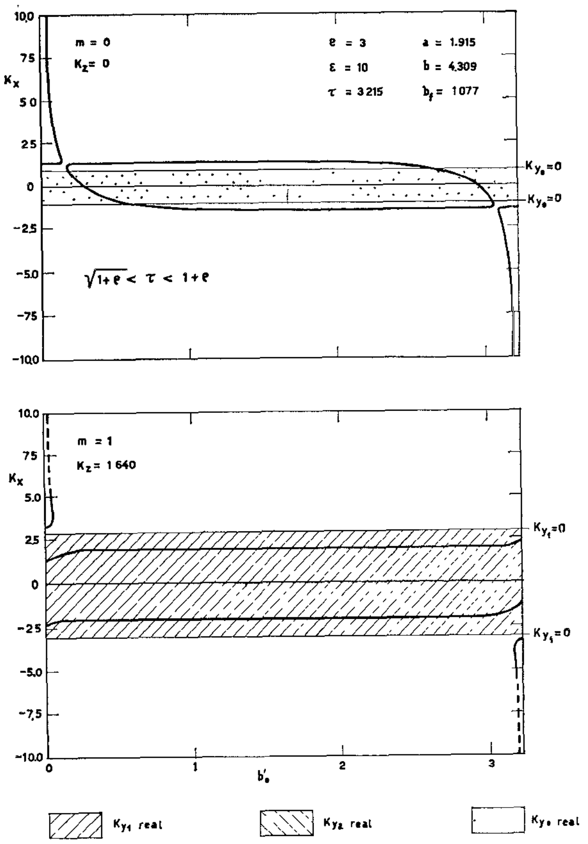


Fig. 2—The mapping of the solutions of the characteristic equation for the indicated values of ρ, ϵ, τ, a, b and b_f. These values may be taken to correspond to

$$M_0 = 0.3\,\frac{Wb}{m^2}; \qquad H_0 = \frac{10^6}{4\pi}\,A/m; \qquad f = \frac{\omega}{2\pi} = 9000\text{ Mc};$$

$$\frac{a}{\omega\sqrt{\mu_0\epsilon_0}} = 10.16\text{ mm}; \qquad \frac{b}{\omega\sqrt{\mu_0\epsilon_0}} = 22.86\text{ mm}; \qquad \frac{b_f}{\omega\sqrt{\mu_0\epsilon_0}} = 5.71\text{ mm}.$$

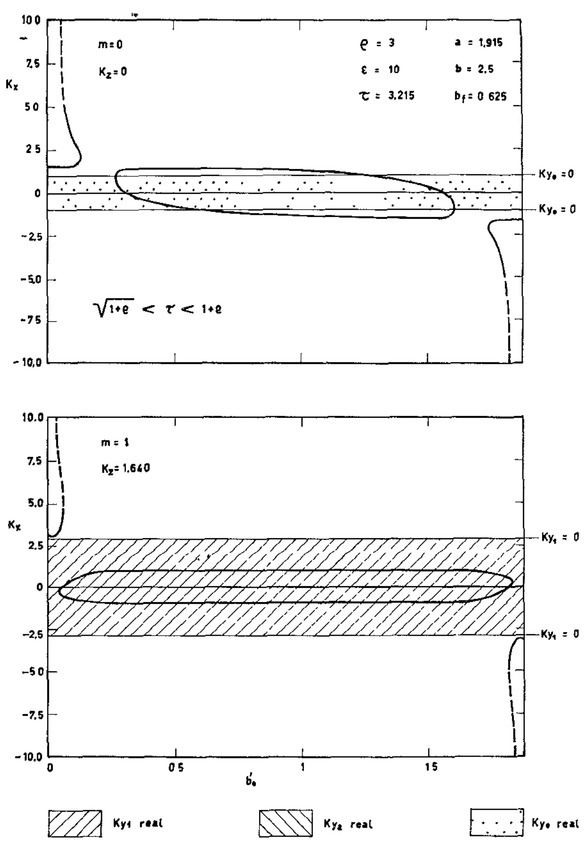


Fig. 3—The same as Fig. 2, except for

$$\frac{b}{\omega\sqrt{\mu_0\epsilon_0}} = 13.25\text{ mm}; \qquad \frac{b_f}{\omega\sqrt{\mu_0\epsilon_0}} = 3.30\text{ mm}.$$

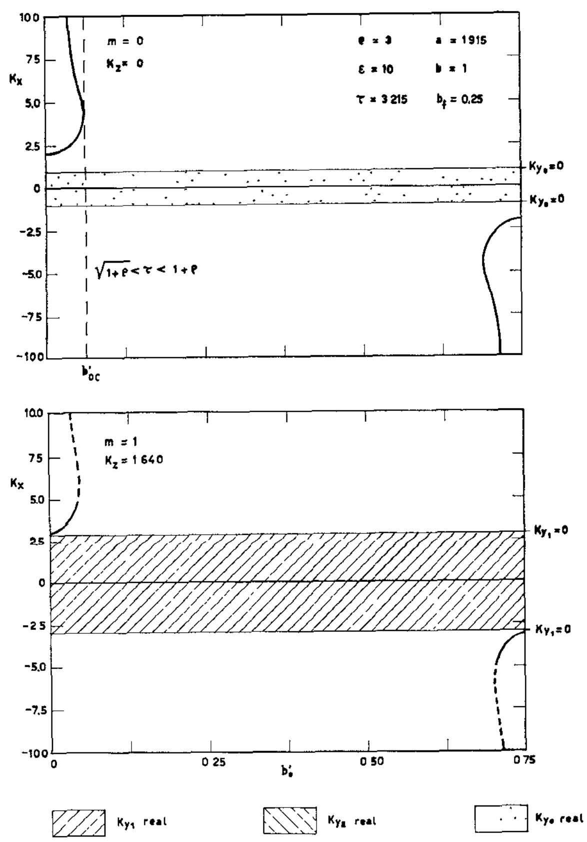
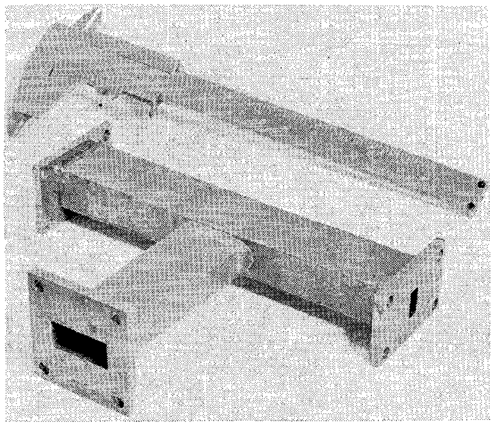
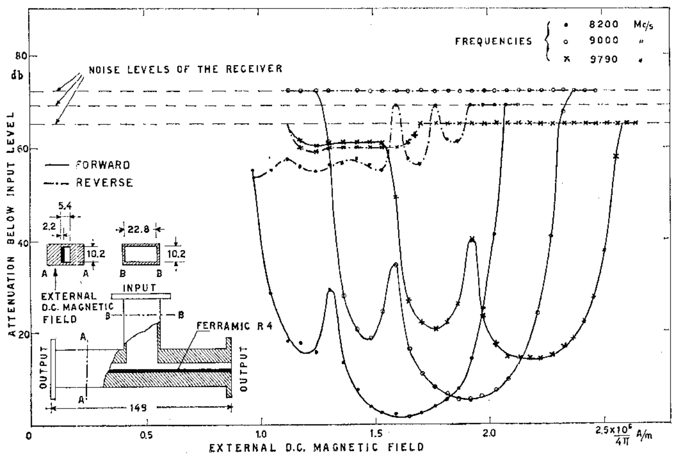


Fig. 4—The same as Fig. 2, except for

$$\frac{b}{\omega\sqrt{\mu_0\epsilon_0}} = 5.30\text{ mm}; \qquad \frac{b_f}{\omega\sqrt{\mu_0\epsilon_0}} = 1.32\text{ mm}.$$

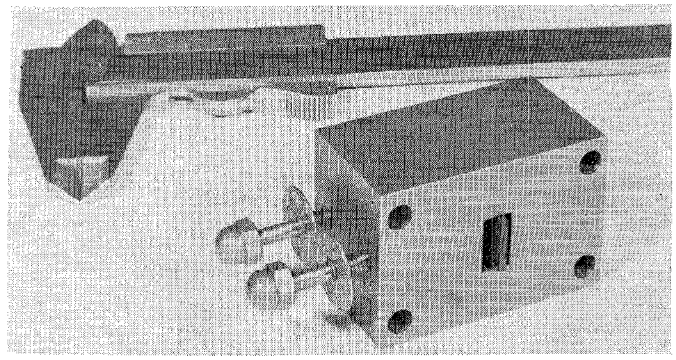


(a)

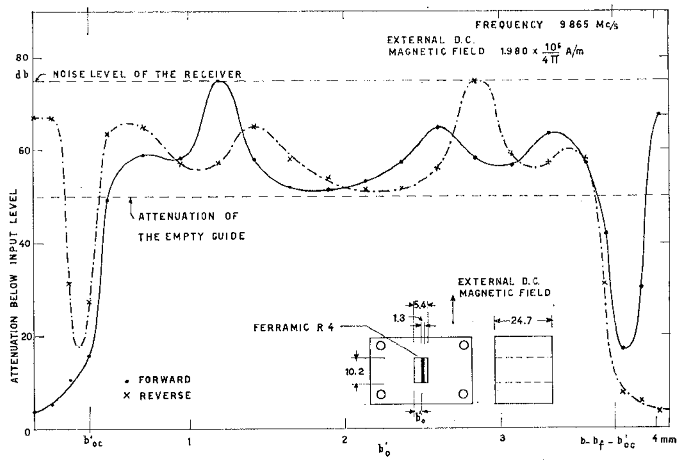


(b)

Fig. 5—(a) T-shaped waveguide built to verify the unidirectional character of a guide of suitable dimensions loaded with magnetized ferrite. (b) Experimental curves of the attenuation below the input level vs the external dc magnetic field for the structure and the frequencies indicated. All the dimensions are in millimeters.



(a)



(b)

Fig. 6—(a) Waveguide section built to investigate the behavior of a guide loaded with a slab of magnetized ferrite. (b) Experimental curves of the attenuation below the input level vs b_0' for the values of the frequency and the external dc magnetic field indicated. All the dimensions are in millimeters.

that, because of the normalization assumed, the group velocity u_g is given by

$$u_g = \frac{1}{\sqrt{\mu_0 \epsilon_0}\,\dfrac{d(\tau k_x)}{d\tau}}.$$

By using the asymptotic expression (7) it is easily seen that by increasing the frequency, *i.e.*, τ, τk_x decreases and vice versa. On the other hand for $b_0'=0$ we have computed τk_x for two values of the frequency slightly above and below the frequency relative to Fig. 4 and we have found that the group velocity is positive. By using these results we can conclude that the two modes corresponding to the same value of b_0' have group velocity of opposite sign. It seems reasonable to extend the validity of this result to modes of the first order.

IV. Experimental

The experiments carried out had two different aims: 1) to prove the unidirectionality of a structure corresponding to the case of Fig. 4 for $b_0'=0$; 2) to verify the general behavior of a structure corresponding to the case of Fig. 4 when b_0' is varied.

For the experiment 1) we have built a T-shaped waveguide [shown in Fig. 5(a)] whose dimensions are given in Fig. 5(b). By sending the RF energy in the input arm we have recorded in Fig. 5(b) the attenuation for both the output arms below the input level vs the external dc magnetic field.[7] This has been done for the three different frequencies indicated in the figure. From the experimental results of Fig. 5(b) the unidirectional character of the structure considered is apparent for a wide band of frequencies.

For the experiment 2) we have built a guide section [shown in Fig. 6(a)] whose dimensions are given in Fig. 6(b). By means of insulating rods and screws the slab of ferrite could be moved parallel to itself inside the guide. By sending energy in the input arm we have recorded in Fig. 6(b) the attenuation below the input level at the output arm vs the distance of the slab of ferrite from one of the walls, for the two opposite values of the dc magnetic field, which correspond to

[7] Note that the dc magnetic field assumed for the theoretical calculations is the internal one, which in the experiments is not uniform and its average is smaller than the external.

forward and reverse propagation. In the figure it is shown the attenuation for the empty guide, *i.e.*, for the same guide without the slab of ferrite.

Let's follow the forward propagation experimental curve from left to right. When the slab of ferrite is against the left side wall the attenuation is at a minimum. By moving the slab of ferrite away from the wall, after a region of low attenuation, the signal goes very rapidly below the level of the attenuation of the empty guide. This is justified by the theoretical results of Fig. 4 and by the considerations about the group velocity. In fact, beyond the distance b_{0c}' no propagation can exist and below such a distance always exists a propagating mode with group velocity in the forward direction.

Moving the slab further away from the left side wall, the signal remains at a level below the empty guide attenuation level until we reach a distance approximately equal to $b-b_f-b_{0c}'$ when the attenuation begins to decrease and then, after it has reached a minimum, increases again above the empty guide attenuation. This last behavior is easily explained by the theoretical results, since beyond the distance $b-b_f-b_{0c}'$, energy begins to pass in the forward direction through modes having positive group velocity. However, since these modes have propagation constants going to $-\infty$ as the slab of ferrite approaches the right-hand side wall and cannot therefore allow propagation in the limit, the minimum of the attenuation experimentally found is explained.

From the preceding discussion we can conclude that there is good agreement between theory and experiments.

Higher-Order Evaluation of Electromagnetic Diffraction by Circular Disks*

W. H. EGGIMANN†

Summary—**The problem of the diffraction of an arbitrary electromagnetic field by a circular perfectly-conducting disk has been solved by using a series representation in powers of $k=2\pi/\lambda$ and the rectangular disk coordinates. The surface current density is given in terms of the field and its derivatives at the center of the disk. General expressions for the electric- and magnetic-dipole moments, the far-field and the scattering coefficient for the case of a plane wave at arbitrary incidence are presented. The calculations agree with results published by other authors. A bibliography of the most recent publications on this problem is included.**

I. Introduction

The problem of the diffraction by a circular conducting disk (or the complementary problem for a circular aperture in an infinite plane conducting screen) has occupied many workers in the field of diffraction theory. The problem can be formulated as follows:

1) the electromagnetic field has to obey Maxwell's equations,
2) the boundary conditions on the surface of the disk have to be fufilled, *e.g.*, for a perfectly-conducting disk the tangential electric field must vanish,
3) the edge conditions [48] at the rim of the disk have to be obeyed; they require that the field energy remains finite, or that the energy density has to be integrable over any finite space. This leads to the requirement that the normal component of the electric field increases not faster than $(1/r)^{1/2}$ where r is the distance from the edge,
4) Sommerfeld's radiation conditions [47] have to be fulfilled.

In this paper a power-series solution in (ka) valid for the small disk problem ($a<\lambda/2\pi$, where $a=$disk radius, $\lambda=$free-space wavelength) and an arbitrary incident field is given. It is essentially an extension of a procedure described by Bouwkamp [45]. The surface current density on the disk up to the third-order approximation in (ka) is calculated in terms of the electromagnetic field and its derivatives at the center of the disk. From these results expressions for the induced electric and magnetic dipole moments and the far-zone fields are derived. The scattering coefficient for a plane wave at arbitrary incidence has been calculated in agreement with formulas given by Lur'e [19] and Kuritsyn [20]. The essential advantage of the expressions obtained in this paper is that they can be used for any primary field. This is important in the case where interaction between several disks is considered. If the spacing between the disks is not large compared with the wavelength, the interaction fields cannot be approximated by a plane wave and the interaction be-

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† Dept. of Elec. Engrg., Case Inst. of Tech., Cleveland, Ohio.